A Maxwell like Formulation of Gravitational Theory in Minkowski Spacetime*

E. A. Notte-Cuello[(1)] and W. A. Rodrigues Jr.[(2)]
[(1)]Departamento de Matemáticas,Universidad de La Serena
Av. Cisternas 1200, La Serena-Chile
[(2)] Institute of Mathematics, Statistics and Scientific Computation
IMECC-UNICAMP CP 6065 13083-859 Campinas, SP, Brazil
walrod@ime.unicamp.br; enotte@userena.cl

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Abstract

In this paper using the Clifford bundle formalism a Lagrangian theory of the Yang-Mills type (with a gauge fixing term and an auto interacting term) for the gravitational field in Minkowski spacetime is presented. It is shown how two simple hypothesis permits the interpretation of the formalism in terms of effective Lorentzian or teleparallel geometries. In the case of a Lorentzian geometry interpretation of the theory the filed equations are shown to be equivalent to Einstein's equations.

1 Introduction

In this paper we present a Lagrangian theory of the gravitational field in Minkowski spacetime[1] $(M \simeq \mathbb{R}^4, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ which is of the Yang-Mills type (containing moreover a gauge fixing term and an auto interaction term related to the vorticity of the fields). In our theory each nontrivial gravitational field configuration can be interpreted as generating an effective Lorentzian spacetime $(M \simeq \mathbb{R}^4, \mathbf{g}, \nabla, \tau_{\mathbf{g}}, \uparrow)$ where $\mathbf{g}$ satisfies Einstein equations or by an effective teleparallel spacetime. Our theory is invariant under diffeomorphisms and under local Lorentz transformations, and is based on *two* assumptions. The first

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[1]Minkowski spacetime will be called Lorentz vacuum, in what follows. Moreover in the pentuple $(M \simeq \mathbb{R}^4, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$, $\boldsymbol{\eta}$ is a Minkowski metric, D is its Levi-Civita connection, $\tau_{\boldsymbol{\eta}}$ is the volume element defining a global orientation and $\uparrow$ refers to a time orientability. The objects in the Loretnzian spacetime structure $(M \simeq \mathbb{R}^4, \mathbf{g}, \nabla, \tau_{\mathbf{g}}, \uparrow)$ have analogous meanings. Details are given in, e.g.,[28, 22].

one is that the gravitational field is a set of Maxwell like fields, which are physical fields in Faraday sense (i.e., of the same ontology as the electromagnetic field, having nothing a priory to do with the geometry of spacetime), which lives in Minkowski spacetime, have its dynamics described by a specified Lagrangian density and couples universally with the matter fields. Such coupling is such that the presence of energy-momentum due to matter fields in some region of Minkowski spacetime distorts the Lorentz vacuum in much the same way that stresses in an elastic body produces deformations in it. This *distortion* permits the introduction of a new metric field $\mathbf{g}$ in M which is the analogous of the Cauchy-Green tensor [6] of elasticity theory. The field $\mathbf{g}$ can be written in terms of the Maxwell like fields (potentials) describing the gravitational field in an appropriate way. Once the Levi-Civita connection ∇ of $\mathbf{g}$ is introduced in the game, it is possible to show that the Maxwell like field equations for the gravitational fields which follows from the variational field implies that $\mathbf{g}$ satisfies Einstein equations. This is done in Section 3. Moreover, it is shown in Section 7 that the formalism can also be interpreted in terms of a $\mathbf{g}$-compatible teleparallel connection in M, which produces in a an clear and elegant way the so called teleparallel equivalent of General Relativity. The situation here is somewhat analogous to the one in the following example. Suppose you have a punctured sphere $\mathring{S}^2$ that lives in $\mathbb{R}^3$. Which is the best geometry that you can use in $\mathring{S}^2$? Well, the answer depends on the applications you have in mind. It may be useful for some problems (computation of curves of minimum length (geodesics)) to use a Riemannian geometrical structure $(\mathring{S}^2, g, \overset{LC}{D})$, where g is pullback on $\mathring{S}^2$ of the Euclidean metric on $\mathbb{R}^3$, and $\overset{LC}{D}$ is the Levi-Civita connection of g, or it may be more useful (e.g., for sailors) to use the structure $(\mathring{S}^2, g, \overset{N}{D})$ where $\overset{N}{D}$ is the Nunes connection (also called navigator, or Columbus connection [22]). There are still some problems [20] where the use of a Euclidean geometry on $\mathring{S}^2$ is the most useful one. This last geometry defines the so called stereographic sphere $(\mathring{S}^2, g', \overset{LC}{D'})$. In it a metric g' is defined by pullback of the Euclidean metric of a tangent plane at the south pole with the diffeomorphism map defined by stereographic projection map (from the north pole). The connection $\overset{LC}{D'}$ is defined as the Levi-Civita connection of g'.

To present the details of our theory we start by introducing $\{\mathbf{x}^\mu\}$, which are global coordinate functions[2] in Einstein-Lorentz-Poincaré gauge for M associated to an arbitrary inertial reference frame[3] $I = \partial/\partial x^0 \in \sec TM$. Let $\{e_\mathbf{a} = \delta^\mu_\mathbf{a} \partial/\partial x^\mu\}$ be an orthonormal basis for TM and $\{\vartheta^\mathbf{a}\}$ the corresponding dual basis for T^*M. We have $\vartheta^\mathbf{a} = \delta^\mathbf{a}_\mu dx^\mu$, $\mathbf{a} = 0, 1, 2, 3$, and we take[4]

[2]The coordinates of $\mathfrak{e} \in M$ in Einstein-Lorentz-Poincaré gauge are $\{x^\mu\} := \{\mathbf{x}^\mu(\mathfrak{e})\}$.

[3]An inertial reference frame satisfies $DI = 0$. See [22] for details.

[4]$\bigwedge^p T^*M$ denotes the bundle of p-forms, $\bigwedge T^*M = \bigoplus_{p=0}^4 \bigwedge^p T^*M$ is the bundle of multiform fields, $\mathcal{C}\ell(M, \eta)$ denotes the Clifford bundle of differential forms. The symbol sec means section. All 'tricks of the trade' necessary for performing the calculations of the present paper are described in [22].

$\vartheta^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$. Of course, we have

$$\boldsymbol{\eta} = \eta_{\mathbf{ab}} \vartheta^{\mathbf{a}} \otimes \vartheta^{\mathbf{b}}. \tag{1}$$

Assumption 1: A non trivial gravitational field is represented by a basis $\{\mathfrak{g}^{\mathbf{a}}\}$ of T^*M, defining a. η-orthonormal coframe bundle for M, which is *not* a coordinate coframe in all M and such that[5] $\eta = \eta_{\mathbf{ab}} \mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$.In what follows we suppose moreover that $\mathfrak{g}^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$. The fields $\mathfrak{g}^{\mathbf{a}}$ in a region of M generated by the matter fields with Lagrangian density $\mathcal{L}_m^M$ are described by a Lagrangian density

$$\mathcal{L} = \mathcal{L}_g^M + \mathcal{L}_m^M, \tag{2}$$

where

$$\mathcal{L}_g^M = -\frac{1}{2} d\mathfrak{g}^{\mathbf{a}} \wedge \underset{\eta}{\star} d\mathfrak{g}_{\mathbf{a}} + \frac{1}{2} \underset{\eta}{\delta} \mathfrak{g}^{\mathbf{a}} \wedge \underset{\eta\eta}{\star\delta} \mathfrak{g}_{\mathbf{a}} + \frac{1}{4} d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}} \wedge \underset{\eta}{\star} d\mathfrak{g}^{\mathbf{b}} \wedge \mathfrak{g}_{\mathbf{b}}, \tag{3}$$

is invariant under local Lorentz transformations[6], which is a kind of gauge freedom, a crucial ingredient of our theory, as will be clear in a while. Moreover, $\underset{\eta}{\star}$ refers to the Hodge dual defined by $\eta = \eta_{\mathbf{ab}} \mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$.

The $\mathfrak{g}^{\mathbf{a}}$ couple universally to the matter fields in such a way that the energy momentum 1-form of the matter fields are given by

$$-\underset{\eta}{\star} T_{\mathbf{a}}^M = \underset{\eta}{\star} T_{\mathbf{a}}^M = \frac{\partial \mathcal{L}_m^M}{\partial \mathfrak{g}^{\mathbf{a}}}. \tag{4}$$

We see that each one of the fields $\mathfrak{g}^{\mathbf{a}}$ in Eq.(3) resembles a potential of an electromagnetic field. Indeed, the first term is of the Yang-Mills type, the second term is a kind of gauging fixing term, for indeed, $\underset{\eta}{\delta} \mathfrak{g}^{\mathbf{a}} = 0$ is analogous to the *Lorenz* condition for the gauge potential of the electromagnetic potential and finally the third term is a self-interacting term, which is proportional to the square of the total 'vorticity' $\Omega = d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}}$ associated to the 1-form fields $\mathfrak{g}^{\mathbf{a}}$. We will derive in Section 4 Maxwell like equations for the gravitational fields. Comparison of our equations with the ones found by [12, 13, 14] are mentioned

We see that in our formulation of the theory of gravitational field there is until now no mention to a Lorentzian spacetime structure $(M \simeq \mathbb{R}^4, \mathbf{g}, \nabla, \tau_{\mathbf{g}}, \uparrow)$. Such structure enters the game by supposing that the most general deformation of the Lorentz vacuum can be described by a diffeomorphism $\mathtt{h} : M \to M$, $\mathfrak{e} \mapsto \mathtt{h}\mathfrak{e}$, and a related gauge metric extensor field h, which are introduced next.

[5]Take notice that $\boldsymbol{\eta} = \eta_{\mathbf{ab}} \vartheta^{\mathbf{a}} \otimes \vartheta^{\mathbf{b}} \neq \eta .{=} \eta_{\mathbf{ab}} \mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$

[6]We observe that various coefficients in Eq.(3) have been selected in order for $\mathcal{L}_g^M$ to be invariant under arbitrary local Lorentz transformations. This means, as the reader may verifiy that under the transformation $\mathfrak{g}^{\mathbf{a}} \mapsto u\mathfrak{g}^{\mathbf{a}}u^{-1}$, $u \in \sec \mathrm{Spin}_{1,3}^e(M,\eta) \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$, $\mathcal{L}_g^M$ is invariant modulo an exact form.

2 $\mathcal{C}\ell(M,\eta)$, $\mathcal{C}\ell(M,\mathtt{g})$, $\mathtt{h}$ and h

2.1 Enter $\mathtt{h}$

Assumption 2: Every physically acceptable gravitational $\{\mathfrak{g}^{\mathbf{a}}\}$ induces a metric field $\mathbf{g} \in \sec T_2^0 M$ which is a Cauchy-Green like tensor [6], i.e., it is the pullback of the metric[7] $\eta = \eta_{\mathbf{ab}} \mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$ under a diffeomorphism $\mathtt{h} : M \to M$, $\mathfrak{e} \mapsto \mathtt{h}\mathfrak{e}$. We have,

$$\mathbf{g} = \mathtt{h}^* \eta = \eta_{\mathbf{ab}} \theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}, \tag{5}$$
$$\mathfrak{g}^{\mathbf{a}} = \mathtt{h}^{*-1} \theta^{\mathbf{a}}. \tag{6}$$

To show that our assumptions imply indeed that $\mathbf{g}$ satisfies Einstein equations as stated above, we need to prove that $\mathcal{L}_g^M$ is equivalent to the Einstein-Hilbert Lagrangian. This will be done after we prove Proposition , which needs some preliminaries.

2.2 Enter h

Consider the Clifford bundles of nonhomogeneous multiform fields $\mathcal{C}\ell(M,\eta)$ and $\mathcal{C}\ell(M,\mathtt{g})$. In $\mathcal{C}\ell(M,\eta)$, where η refers to the standard metric on the cotangent bundle associated to $\boldsymbol{\eta} = \eta_{\mathbf{ab}} \vartheta^{\mathbf{a}} \otimes \vartheta^{\mathbf{b}}$, the Clifford product will be denoted by juxtaposition of symbols, the scalar product by $\cdot$ and the contractions by $\lrcorner$ and $\llcorner$ and by $\star$ we denote the Hodge dual. The Clifford product in $\mathcal{C}\ell(M,\mathtt{g})$ will be denoted by the symbol $\vee$, the scalar product will be denoted by $\bullet$ and the contractions by $\underset{\mathbf{g}}{\lrcorner}$ and $\underset{\mathbf{g}}{\llcorner}$ while by $\underset{\mathbf{g}}{\star}$ we denote the Hodge dual operator associated to $\mathbf{g}$.

Let $\{\mathbf{e}_a\}$ be a non coordinate basis of TM dual to the cobasis $\{\theta^{\mathbf{a}}\}$. We take the $\theta^{\mathbf{a}}$ as sections of the Clifford bundle $\mathcal{C}\ell(M,\eta)$, i.e., $\theta^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$. In this basis where according to Eq.(5) $\mathbf{g} = \eta_{\mathbf{ab}} \theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}$ we have that

$$\eta = g_{\mathbf{ab}} \theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}, \tag{7}$$

and moreover $\mathtt{g} \in \sec T_0^2 M$ is given by

$$\mathtt{g} = \eta^{\mathbf{ab}} \mathbf{e}_{\mathbf{a}} \otimes \mathbf{e}_{\mathbf{b}}. \tag{8}$$

The cobasis $\{\vartheta^{\mathbf{a}}\}$ defines a Clifford product in $\mathcal{C}\ell(M,\eta)$ by

$$\vartheta^{\mathbf{a}} \vartheta^{\mathbf{b}} + \vartheta^{\mathbf{b}} \vartheta^{\mathbf{a}} = 2\eta^{\mathbf{ab}}, \tag{9}$$

and taking into account that the cobasis $\{\theta^{\mathbf{a}}\}$ defines a *deformed* Clifford product $\vee$ in $\mathcal{C}\ell(M,\eta)$ (see details in [22, 19]) generating a representation of the

[7] Take notice that $\boldsymbol{\eta} = \eta_{\mathbf{ab}} \vartheta^{\mathbf{a}} \otimes \vartheta^{\mathbf{b}} \neq \eta . = \eta_{\mathbf{ab}} \mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$

Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$ we can write

$$\theta^{\mathbf{a}} \vee \theta^{\mathbf{b}} = \theta^{\mathbf{a}} \bullet \theta^{\mathbf{b}} + \theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}},$$
$$\theta^{\mathbf{a}} \vee \theta^{\mathbf{b}} + \theta^{\mathbf{b}} \vee \theta^{\mathbf{a}} = 2\eta^{\mathbf{ab}}. \tag{10}$$

Then, as proved, e.g., in [22, 19] there exist $(1,1)$-extensor fields g and h such that

$$\mathtt{g}(\theta^{\mathbf{a}}, \theta^{\mathbf{b}}) = \theta^{\mathbf{a}} \bullet \theta^{\mathbf{b}} = \theta^{\mathbf{a}} \cdot g(\theta^{\mathbf{b}}) = h(\theta^{\mathbf{a}}) \cdot h(\theta^{\mathbf{b}}) = \eta^{\mathbf{ab}}. \tag{11}$$

The gauge metric extensor $h : \sec \bigwedge^1 T^*M \to \sec \bigwedge^1 T^*M$ is defined by

$$h(\theta^{\mathbf{a}}) = \vartheta^{\mathbf{a}}. \tag{12}$$

2.3 Relation Between h and $\mathtt{h}^*$

Recall that **Assumption 2** says that every physically acceptable $\mathbf{g}$ is a Cauchy-Green like tensor [6], i.e., it is the pullback of the metric $\eta = \eta_{\mathbf{ab}} \mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$ under a diffeomorphism $\mathtt{h} : M \to M$, $\mathfrak{e} \mapsto \mathtt{h}\mathfrak{e}$.

Introduce Riemann normal coordinates functions $\{\boldsymbol{y}^\mu\}$ for M such that

$$\mathbf{y}^\mu(\mathtt{h}\mathfrak{e}) = y^\mu, \tag{13}$$

and being $\mathbf{x}^\mu$ the coordinates in the the Einstein-Lorentz-Poincaré gauge for M already introduced and obeying $\mathbf{x}^\mu(\mathfrak{e}) = x^\mu$, we have

$$\eta|_{\mathtt{h}\mathfrak{e}} = \eta_{\mathbf{ab}} \delta^{\mathbf{a}}_\mu \delta^{\mathbf{b}}_\nu dy^\mu \otimes dy^\nu, \;\; \mathbf{g}|_{\mathfrak{e}} = g_{\mu\nu} dx^\mu \otimes dx^\nu. \tag{14}$$

Moreover, let $y^\mu = \mathtt{h}^\mu(x^\nu)$ be the coordinate expression[8] for $\mathtt{h}$. Since

$$\mathtt{h}^*\eta(\delta^\mu_{\mathbf{a}} \partial/\partial x^\mu, \delta^\nu_{\mathbf{b}} \partial/\partial x^\nu)|_{\mathfrak{e}} = \eta(\delta^\mu_{\mathbf{a}} \mathtt{h}_* \partial/\partial x^\mu, \delta^\nu_{\mathbf{b}} \mathtt{h}_* \partial/\partial x^\nu)|_{\mathtt{h}\mathfrak{e}},$$

$$\mathbf{g} = \mathtt{h}^*\eta = \eta_{\mathbf{ab}} \delta^{\mathbf{a}}_\alpha \delta^{\mathbf{b}}_\beta \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} dx^\mu \otimes dx^\nu \tag{15}$$

with

$$g_{\mu\nu} = \eta_{\mathbf{ab}} \delta^{\mathbf{a}}_\alpha \delta^{\mathbf{b}}_\beta \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}. \tag{16}$$

Now, take notice that at $\mathfrak{e}$, $\{\mathbf{f_a}\}$, $\mathbf{f_a} = \delta^\mu_{\mathbf{a}} \mathtt{h}^{-1}_* \partial/\partial y^\mu = \delta^\mu_{\mathbf{a}} \frac{\partial x^\nu}{\partial y^\mu} \frac{\partial}{\partial x^\nu}$ is not (in general) a coordinate basis for TM. It is also not η-orthonormal[9]. The dual basis of $\{\mathbf{f_a}\}$ at $\mathfrak{e}$ is $\{\sigma^{\mathbf{a}}|_{\mathfrak{e}}\}$, with $\sigma^{\mathbf{a}}|_{\mathfrak{e}} = \delta^{\mathbf{a}}_\mu \frac{\partial y^\mu}{\partial x^\nu} dx^\nu \Big|_{\mathfrak{e}} = \mathtt{h}^*(\delta^{\mathbf{a}}_\mu dy^\mu \big|_{\mathtt{h}\mathfrak{e}})$. Then it exists an extensor field $\check{h}$ differing from h by a Lorentz extensor, i.e., $\check{h} = h\Lambda$ such that $\sigma^{\mathbf{a}}|_{\mathfrak{e}} = \check{h}^{-1}(\delta^{\mathbf{a}}_\mu dy^\mu)\big|_{\mathfrak{e}} = \check{h}^{-1\mathbf{a}}_\mu dy^\mu\big|_{\mathfrak{e}}$, we have for any $\mathfrak{e} \in M$,

$$\delta^{\mathbf{a}}_\alpha \frac{\partial y^\alpha}{\partial x^\mu} = \check{h}^{-1\mathbf{a}}_\mu. \tag{17}$$

[8] Recall that the $\mathtt{h}^\mu$ are assumed invertible differentiable functions.

[9] Indeed, $\eta(\mathbf{e_a}, \mathbf{e_b}) = \delta^\mu_{\mathbf{a}} \delta^\nu_{\mathbf{b}} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \eta_{\alpha\beta}$.

To determine $\check{h}$ we proceed as follows. Suppose $\mathbf{g} = \eta_{\mathbf{ab}}\sigma^{\mathbf{a}} \otimes \sigma^{\mathbf{b}}$ is known. Let (v_i, λ_i) be respectively the eigen-covectors and the eigenvalues of g, i.e., $g(v_i) = \lambda_i v_i$ (no sum in i) and $\{\vartheta^{\mathbf{a}}\}$ the η-orthonormal coordinate basis for T^*M introduced above. Then, since $g = \check{h}^\dagger \check{h}$ we immediately have

$$\check{h}(v_i) = \sqrt{|\lambda_i|}\eta(v_i, \vartheta_{\mathbf{a}})\vartheta^{\mathbf{a}}, \tag{18}$$

which then determines the extensor field h (modulus a local Lorentz rotation) at any spacetime point, and thus the diffeomorphism $\mathtt{h}$ (modulus a local Lorentz rotation).

3 Enter ∇

Now, any other $\mathbf{g}$-orthonormal non coordinate cobasis $\{\theta^{\mathbf{a}}\}$ is related by a Lorentz extensor field Λ, (i.e., $\Lambda^\dagger g \Lambda = g$) to the cobasis $\{\sigma^{\mathbf{a}}\}$ by $\theta^{\mathbf{a}} = \Lambda(\sigma^{\mathbf{a}}) = U\sigma^{\mathbf{a}}U^{-1}$, $U \in \sec \mathrm{Spin}^e_{1,3}(M, \mathtt{g}) \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$. Then, $\mathbf{g} = \eta_{\mathbf{ab}}\sigma^{\mathbf{a}} \otimes \sigma^{\mathbf{b}} = \eta_{\mathbf{ab}}\theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}$ and **assumption 2** says that $\mathbf{g} = \eta_{\mathbf{ab}}\theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}} = \mathtt{h}^*\eta = \mathtt{h}^*(\eta_{\mathbf{ab}}\mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}})$, i.e., $\mathfrak{g}^{\mathbf{a}} = \mathtt{h}^{*-1}\theta^{\mathbf{a}}$. Taking into account also that $\mathfrak{g}_{\mathbf{a}} = \eta_{\mathbf{ab}}\mathfrak{g}^{\mathbf{b}}$, we can write using the identities

$$\begin{aligned}
d\mathtt{h}^*K &= \mathtt{h}^*dK \\
\underset{\mathbf{g}}{\star}d\underset{\mathbf{g}}{\star}\mathtt{h}^*K &= \mathtt{h}^*\underset{\eta}{\star}d\underset{\eta}{\star}K, \\
\mathtt{h}^*L\underset{\mathbf{g}}{\lrcorner}\mathtt{h}^*K &= \mathtt{h}^*(L\underset{\eta}{\lrcorner}K),
\end{aligned} \tag{19}$$

valid for any $L \in \sec \bigwedge^r T^*M$, $K \in \sec \bigwedge^p T^*M$, $r \leq p$, that

$$\begin{aligned}
d\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}d\theta_{\mathbf{a}} &= \mathtt{h}^*d\mathtt{h}^{*-1}\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}d\theta_{\mathbf{a}} \\
&= \mathtt{h}^*d\mathtt{h}^{*-1}\theta^{\mathbf{a}} \wedge \mathtt{h}^*\underset{\eta}{\star}d\mathtt{h}^{*-1}\theta_{\mathbf{a}} \\
&\mathtt{h}^*(d\mathtt{h}^{*-1}\theta^{\mathbf{a}} \wedge \underset{\eta}{\star}d\mathtt{h}^{*-1}\theta_{\mathbf{a}}) \\
&= \mathtt{h}^*(d\mathfrak{g}^{\mathbf{a}} \wedge \underset{\eta}{\star}d\mathfrak{g}_{\mathbf{a}})
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\underset{\eta}{\delta}\theta^{\mathbf{a}} \wedge \underset{\eta\eta}{\star\delta}\theta_{\mathbf{a}} &= \mathtt{h}^*(\underset{\eta}{\delta}\mathfrak{g}^{\mathbf{a}} \wedge \underset{\eta\eta}{\star\delta}\mathfrak{g}_{\mathbf{a}}), \\
d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}} &= \mathtt{h}^*(d\mathfrak{g}^{\mathbf{a}} \wedge \mathfrak{g}_{\mathbf{a}} \wedge \underset{\eta}{\star}d\mathfrak{g}^{\mathbf{b}} \wedge \mathfrak{g}_{\mathbf{b}}).
\end{aligned} \tag{21}$$

Then, the Lagragian for the gravitational field becomes

$$\mathcal{L}_g^M = \mathtt{h}^{*-1}\mathcal{L}_g, \tag{22}$$

where

$$\mathcal{L}_g = -\frac{1}{2} d\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}} + \frac{1}{2} \underset{\mathbf{g}}{\delta} \theta^{\mathbf{a}} \wedge \underset{\mathbf{gg}}{\star\delta} \theta_{\mathbf{a}} + \frac{1}{4} \left(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}} \right) \wedge \underset{\mathbf{g}}{\star} \left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}} \right). \tag{23}$$

We now show that $\mathcal{L}_g$ differs from the Einstein-Hilbert Lagrangian by an exact differential[10]. It is at this point that the Levi-Civita connection ∇ of $\mathbf{g}$ comes to play. Indeed, we introduce the connections 1-forms $\omega^{\mathbf{a}}_{\mathbf{b}}$ and the curvature 2-forms in the cobasis $\{\theta^{\mathbf{a}}\}$ of the Lorentzian spacetime structure $(M \simeq \mathbb{R}^4, \mathbf{g}, \nabla, \tau_{\mathbf{g}}, \uparrow)$ through Cartan's structure equations

$$d\theta^{\mathbf{a}} + \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} = 0, \tag{24}$$

$$\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} = d\omega^{\mathbf{a}}_{\mathbf{b}} + \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}}. \tag{25}$$

Next, we suppose that all objects in Eq.(24) and Eq.(25) are forms in $\mathcal{C}\ell(M, \mathtt{g})$ represented as explained above in $\mathcal{C}\ell(M, \eta)$. Under this condition Eq.(24) can be easily inverted, i.e., we get

$$\omega^{\mathbf{cd}} = \frac{1}{2} \left[\theta^{\mathbf{d}} \underset{\mathbf{g}}{\lrcorner} d\theta^{\mathbf{c}} - \theta^{\mathbf{c}} \underset{\mathbf{g}}{\lrcorner} d\theta^{\mathbf{d}} + \theta^{\mathbf{c}} \underset{\mathbf{g}}{\lrcorner} \left(\theta^{\mathbf{d}} \underset{\mathbf{g}}{\lrcorner} d\theta_{\mathbf{a}} \right) \theta^{\mathbf{a}} \right]. \tag{26}$$

Remark 1 *It is crucial to observe that the Levi-Civita connection* ∇ *of* $\mathbf{g}$ *such that* $\nabla_{\mathbf{e_a}} \theta^{\mathbf{b}} = -L^{\mathbf{b}}_{\mathbf{ac}} \theta^{\mathbf{c}}$, $\omega^{\mathbf{b}}_{\mathbf{a}} = -L^{\mathbf{b}}_{\mathbf{ac}} \theta^{\mathbf{c}}$ *is not (of course) the pullback of the Levi-Civita connection* D *of* $\boldsymbol{\eta}$ *where* $D_{\mathbf{e_a}} \theta^{\mathbf{b}} = -\Gamma^{\mathbf{b}}_{\mathbf{ac}} \theta^{\mathbf{c}}$. *Indeed, as the reader may easily verify, if that was the case the Riemann tensor of* ∇ *would be null. Recall that since we supposed that* $d\mathfrak{g}^{\mathbf{a}} = -\frac{1}{2} c'^{\mathbf{a}}_{\mathbf{kb}} \mathfrak{g}^{\mathbf{k}} \wedge \mathfrak{g}^{\mathbf{b}} \neq 0$, *we can introduce on* M *connection 1-forms* $\omega'^{\mathbf{a}}_{\mathbf{b}} := \frac{1}{2} c'^{\mathbf{a}}_{\mathbf{kb}} \mathfrak{g}^{\mathbf{k}}$ *such that they define a connection* ∇' *wich is metric compatible with* $\eta = \eta_{\mathbf{ab}} \mathfrak{g}^{\mathbf{a}} \otimes \mathfrak{g}^{\mathbf{b}}$, $(\nabla'_{\mathfrak{e}_{\mathbf{a}}} \eta = 0)$, *for which the torsion tensor is* $\Theta'^{\mathbf{a}} = d\mathfrak{g}^{\mathbf{a}} + \omega'^{\mathbf{a}}_{\mathbf{b}} \wedge \mathfrak{g}^{\mathbf{b}} = 0$ *and* $\mathcal{R}'^{\mathbf{a}}_{\mathbf{b}} = d\omega'^{\mathbf{a}}_{\mathbf{b}} + \omega'^{\mathbf{a}}_{\mathbf{c}} \wedge \omega'^{\mathbf{c}}_{\mathbf{b}} \neq 0$. *Now,* ∇ *can be wieved as the pullback of* ∇' *and of course, we have that the connection 1-forms of* ∇ *are* $\omega^{\mathbf{a}}_{\mathbf{b}} = \mathtt{h}^* \omega'^{\mathbf{a}}_{\mathbf{b}} = \frac{1}{2} c'^{\mathbf{a}}_{\mathbf{kb}} \theta^{\mathbf{k}}$ *and* $\nabla \mathbf{g} = 0$. *Moreover,* $\Theta^{\mathbf{a}} = \mathtt{h}^* \Theta'^{\mathbf{a}} = 0$, *but* $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} = \mathtt{h}^* \mathcal{R}'^{\mathbf{a}}_{\mathbf{b}} \neq 0$.

3.1 Relation Between $\mathcal{L}_{EH}$ and $\mathcal{L}_g$

Recall that the classical Einstein-Hilbert Lagrangian in appropriate (geometrical) units is

$$\mathcal{L}_{EH} = \frac{1}{2} R \tau_{\mathbf{g}}, \tag{27}$$

where $R = \eta^{\mathbf{cd}} R_{\mathbf{cd}}$ is the scalar curvature. Now, observe that we can write $\mathcal{L}_{EH}$ as

$$\mathcal{L}_{EH} = \frac{1}{2} \mathcal{R}_{\mathbf{cd}} \wedge \underset{\mathbf{g}}{\star} (\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}), \tag{28}$$

[10] Observe that $-\frac{1}{2}(d\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}} - \underset{\mathbf{g}}{\delta} \theta^{\mathbf{a}} \wedge \underset{\mathbf{gg}}{\star\delta} \theta_{\mathbf{a}}) = \frac{1}{4} (d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \wedge \underset{\mathbf{g}}{\star} \left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{a}} \right)$ and $-\frac{1}{2} (d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{b}}) \wedge \underset{\mathbf{g}}{\star} \left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{a}} \right) + \frac{1}{4} (d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}) \wedge \underset{\mathbf{g}}{\star} \left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}} \right)$ is known to differ from the Einstein-Hilbert Lagrangian by an exact differential[26, 21]

where

$$\mathcal{R}^{\mathbf{a}}_{\mathbf{d}} = d\omega^{\mathbf{a}}_{\mathbf{b}} + \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}}, \tag{29}$$

are the curvature 2-forms. Indeed, using well known identities (see, e.g., Chapter 2 of [22]), we have

$$\begin{aligned}\mathcal{R}_{\mathbf{cd}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) &= (\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) \wedge \underset{\mathbf{g}}{\star}\mathcal{R}_{\mathbf{cd}} = -\theta^{\mathbf{c}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{d}} \underset{\mathbf{g}}{\lrcorner} \mathcal{R}_{\mathbf{cd}}) \\ &= -\underset{\mathbf{g}}{\star}[\theta^{\mathbf{c}} \underset{\mathbf{g}}{\lrcorner}(\theta^{\mathbf{d}} \underset{\mathbf{g}}{\lrcorner} \mathcal{R}_{\mathbf{cd}})],\end{aligned} \tag{30}$$

and since

$$\begin{aligned}\theta^{\mathbf{d}} \underset{\mathbf{g}}{\lrcorner} \mathcal{R}_{\mathbf{cd}} &= \frac{1}{2} R_{\mathbf{cdab}} \theta^{\mathbf{d}} \underset{\mathbf{g}}{\lrcorner} (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) = \frac{1}{2} R_{\mathbf{cdab}} (\eta^{\mathbf{da}} \theta^{\mathbf{b}} - \eta^{\mathbf{db}} \theta^{\mathbf{a}}) \\ &= -R_{\mathbf{ca}} \theta^{\mathbf{b}} = -\mathcal{R}_{\mathbf{c}},\end{aligned} \tag{31}$$

it follows that $-\theta^{\mathbf{c}} \underset{\mathbf{g}}{\lrcorner}(\theta^{\mathbf{d}} \underset{\mathbf{g}}{\lrcorner} \mathcal{R}_{\mathbf{cd}}) = \theta^{\mathbf{c}} \bullet \mathcal{R}_c = R$. The $\mathcal{R}_{\mathbf{c}}$, are called the Ricci 1-forms.

Next we establish the following proposition.

Proposition 2 *The Einstein-Hilbert Lagrangian can be written as*

$$\mathcal{L}_{EH} = -d\left(\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}}\right) + \mathcal{L}_g, \tag{32}$$

where

$$\mathcal{L}_g = -\frac{1}{2} \tau_{\mathbf{g}} \theta^{\mathbf{c}} \underset{\mathbf{g}}{\lrcorner} \theta^{\mathbf{b}} \underset{\mathbf{g}}{\lrcorner} (\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}}), \tag{33}$$

is the first order Lagrangian density (first introduced by Einstein)
Proof. *That $\mathcal{L}_g$ is the intrinsic form of the Einstein first order Lagrangian in the gauge defined by $\theta^{\mathbf{a}}$ is easily seen writing $\omega^{\mathbf{a}}_{\mathbf{b}} = L^{\mathbf{a}}_{\mathbf{bc}} \theta^{\mathbf{c}}$. Indeed, we immediately verify using again well known identities (see, e.g., Chapter 2 of [22]), which give*

$$\theta^{\mathbf{c}} \underset{\mathbf{g}}{\lrcorner} \theta^{\mathbf{b}} \underset{\mathbf{g}}{\lrcorner} (\omega_{\mathbf{ac}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}}) = \eta^{\mathbf{bk}} \left(L^{\mathbf{d}}_{\mathbf{kc}} L^{\mathbf{c}}_{\mathbf{db}} - L^{\mathbf{d}}_{\mathbf{dc}} L^{\mathbf{c}}_{\mathbf{kb}} \right). \tag{34}$$

To prove that $\mathcal{L}_{EH}$ can be written as in Eq.(32) we start using Cartan's second structure equation to write Eq.(28) as:

$$\begin{aligned}\mathcal{L}_{EH} &= \frac{1}{2} d\omega_{\mathbf{ab}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + \frac{1}{2} \omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \\ &= \frac{1}{2} d[\omega_{\mathbf{ab}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] + \frac{1}{2} \omega_{\mathbf{ab}} \wedge \underset{\mathbf{g}}{\star} d(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + \frac{1}{2} \omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) \\ &= \frac{1}{2} d[\omega_{\mathbf{ab}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] - \frac{1}{2} \omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}).\end{aligned} \tag{35}$$

Next, using again well known identities (see ,e.g., Chapter 2 of [22]) we get (recall that $\omega^{\mathbf{cd}} = -\omega^{\mathbf{dc}}$)

$$\begin{aligned}(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) \underset{\mathbf{g}}{\wedge} \star\, \omega_{\mathbf{cd}} &= -\underset{\mathbf{g}}{\star}[\omega^{\mathbf{cd}} \underset{\mathbf{g}}{\lrcorner} (\theta_{\mathbf{c}} \wedge \theta_{\mathbf{d}})] \\ &= \underset{\mathbf{g}}{\star}[(\omega^{\mathbf{cd}} \underset{\mathbf{g}}{\cdot} \theta_{\mathbf{d}}) \theta_{\mathbf{c}} - (\omega^{\mathbf{cd}} \underset{\mathbf{g}}{\cdot} \theta_{\mathbf{c}}) \theta_{\mathbf{d}}] = 2 \underset{\mathbf{g}}{\star}[(\omega^{\mathbf{cd}} \underset{\mathbf{g}}{\cdot} \theta_{\mathbf{d}}) \theta_{\mathbf{c}}],\end{aligned} \tag{36}$$

and from Cartan's first structure equation we have ■

Proof.

$$\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}} = \theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}(\omega_{\mathbf{ba}} \wedge \theta^{\mathbf{b}}) = -\underset{\mathbf{g}}{\star}[\theta_{\mathbf{a}} \underset{\mathbf{g}}{\lrcorner}(\omega^{\mathbf{ba}} \wedge \theta_{\mathbf{b}})]$$
$$= -\underset{\mathbf{g}}{\star}[(\theta_{\mathbf{a}} \underset{\mathbf{g}}{\cdot} \omega^{\mathbf{ba}})\theta_{\mathbf{b}}] = -\underset{\mathbf{g}}{\star}[(\theta^{\mathbf{a}} \underset{\mathbf{g}}{\cdot} \omega_{\mathbf{ba}})\theta_{\mathbf{b}}], \tag{37}$$

from where it follows that

$$\frac{1}{2} d[(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) \wedge \underset{\mathbf{g}}{\star}\omega_{\mathbf{cd}}] = -d(\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}}]). \tag{38}$$

On the other hand the second term in the last line of Eq.(35) can be written as

$$\frac{1}{2}\omega_{\mathbf{ab}} \wedge \omega_{\mathbf{c}}^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}})$$
$$= -\frac{1}{2}\underset{\mathbf{g}}{\star}[(\theta^{\mathbf{b}} \underset{\mathbf{g}}{\cdot} \omega_{\mathbf{ab}})(\theta^{\mathbf{c}} \underset{\mathbf{g}}{\cdot} \omega_{\mathbf{c}}^{\mathbf{a}}) - (\theta^{\mathbf{b}} \underset{\mathbf{g}}{\cdot} \omega_{\mathbf{c}}^{\mathbf{a}})(\theta^{\mathbf{c}} \underset{\mathbf{g}}{\cdot} \omega_{\mathbf{ab}})].$$

Now,

$$(\theta^{\mathbf{b}} \underset{\mathbf{g}}{\cdot} \omega_{\mathbf{ab}})(\theta^{\mathbf{c}} \underset{\mathbf{g}}{\cdot} \omega_{\mathbf{c}}^{\mathbf{a}})$$
$$= \omega_{\mathbf{ab}} \underset{\mathbf{g}}{\cdot} [(\theta^{\mathbf{c}} \underset{\mathbf{g}}{\cdot} \omega_{\mathbf{c}}^{\mathbf{a}})\theta^{\mathbf{b}}]$$
$$= \omega_{\mathbf{ab}} \underset{\mathbf{g}}{\cdot} [\theta^{\mathbf{c}} \underset{\mathbf{g}}{\lrcorner}(\omega_{\mathbf{c}}^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + \omega^{\mathbf{ab}}]$$
$$= (\omega_{\mathbf{ab}} \wedge \theta^{\mathbf{c}}) \underset{\mathbf{g}}{\lrcorner}(\omega_{\mathbf{c}}^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + \omega_{\mathbf{cd}} \underset{\mathbf{g}}{\cdot} \omega^{\mathbf{cd}}]$$

and taking into account that $d\theta^{\mathbf{a}} = -\omega_{\mathbf{b}}^{\mathbf{a}} \wedge \theta^{\mathbf{b}}$, $d \underset{\mathbf{g}}{\star} \theta^{\mathbf{a}} = -\omega_{\mathbf{b}}^{\mathbf{a}} \underset{\mathbf{g}}{\cdot} \theta^{\mathbf{b}}$ and that $\delta\theta_{\mathbf{a}} = -\underset{\mathbf{g}}{\star}^{-1} d \underset{\mathbf{g}}{\star} \theta_{\mathbf{a}}$ we have

$$\frac{1}{2}\omega_{\mathbf{ab}} \wedge \omega_{\mathbf{c}}^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) = \frac{1}{2}[-d\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}} - \underset{\mathbf{g}}{\delta}\theta^{\mathbf{a}} \wedge \underset{\mathbf{gg}}{\star\delta}\theta_{\mathbf{a}} + \omega_{\mathbf{cd}} \wedge \underset{\mathbf{g}}{\star}\omega^{\mathbf{cd}}] \tag{39}$$

Next, using Eq.(26) the last term in the last equation can be written as

$$\frac{1}{2}\omega_{\mathbf{cd}} \wedge \underset{\mathbf{g}}{\star}\omega^{\mathbf{cd}} = d\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}} - \frac{1}{4}(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}) \wedge \underset{\mathbf{g}}{\star}\left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}}\right)$$

and we finally get

$$\mathcal{L}_g = -\frac{1}{2} d\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}} + \frac{1}{2}\underset{\mathbf{g}}{\delta}\theta^{\mathbf{a}} \wedge \underset{\mathbf{gg}}{\star\delta}\theta_{\mathbf{a}} + \frac{1}{4}(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}}) \wedge \underset{\mathbf{g}}{\star}\left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}}\right), \tag{40}$$

and the proposition is proved. ■

Now, since
$$d(\theta^{\mathbf{a}}\wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}}) = \mathtt{h}^{*}d(\mathfrak{g}^{\mathbf{a}}\wedge \underset{\eta}{\star} d\mathfrak{g}_{\mathbf{a}})$$
we see taking into account Eq.(22) that[11]
$$\mathcal{L}_{EH} = -d(\theta^{\mathbf{a}}\wedge \underset{\mathbf{g}}{\star} d\theta_{\mathbf{a}}) + \mathcal{L}_{g} = \mathtt{h}^{*}[-d(\mathfrak{g}^{\mathbf{a}}\wedge \underset{\eta}{\star} d\mathfrak{g}_{\mathbf{a}}) + \mathcal{L}_{g}^{M}] = \mathtt{h}^{*}\mathcal{L}_{EH}^{M} \tag{41}$$

Since the variational principle $\boldsymbol{\delta}\int \mathcal{L}_{EH} = 0$ implies $\boldsymbol{\delta}\int \mathtt{h}^{*-1}\mathcal{L}_{EH} = 0$ and thus $\boldsymbol{\delta}\int \mathcal{L}_{EH}^{M} = 0$ we just proved that as stated $\mathcal{L}_{EH}$ and $\mathcal{L}_{EH}^{M}$ are indeed equivalent.

4 Maxwell like Form of the Gravitational Equations

Call $\mathfrak{F}^{\mathbf{a}} = d\mathfrak{g}^{\mathbf{a}}$. Let us verify that the $\mathfrak{F}^{\mathbf{a}}$ satisfy Maxwell (like) equations, i.e., when the energy-momentum tensor of matter the fields is non zero we have in general
$$d\mathfrak{F}^{\mathbf{a}} = 0,\ \underset{\eta}{\delta}\mathfrak{F}^{\mathbf{a}} = J_{\mathbf{m}}^{\mathbf{a}}, \tag{42}$$
where $J_{\mathbf{m}}^{\mathbf{a}}$ is an appropriate current which we now determine.

Since $\underset{\eta}{\delta}\mathfrak{F}^{\mathbf{a}} = \mathtt{h}^{*-1}\underset{\mathbf{g}}{\delta} d\theta^{\mathbf{a}}$ we can write remembering the definition of the Hodge Laplacian
$$-\mathtt{h}^{*-1}\underset{\mathbf{g}}{\delta} d\theta^{\mathbf{a}} = -\mathtt{h}^{*-1}(\underset{\mathbf{g}}{\delta} d\theta^{\mathbf{a}} + d\underset{\mathbf{g}}{\delta}\theta^{\mathbf{a}}) + \mathtt{h}^{*-1}d\underset{\mathbf{g}}{\delta}\theta^{\mathbf{a}} = \mathtt{h}^{*-1}\lozenge\theta^{\mathbf{a}} + \mathtt{h}^{*-1}d\underset{\mathbf{g}}{\delta}\theta^{\mathbf{a}} \tag{43}$$

Now, recall from [23, 22] that the Hodge Laplacian can be written as the square of the Dirac operator $\boldsymbol{\partial} = \theta^{\mathbf{c}}\nabla_{\mathbf{e_c}}$ associated to ∇, the Levi-Civita connection of $\mathbf{g}$,i.e., for any $K \in \sec \bigwedge^{p} T^{*}M \hookrightarrow \mathcal{C}\ell(M,\mathtt{g})$,
$$\lozenge K = -\boldsymbol{\partial}^{2}K = (\boldsymbol{\partial}\wedge\boldsymbol{\partial})K + (\boldsymbol{\partial}\underset{\mathbf{g}}{\cdot}\boldsymbol{\partial})K,$$
where $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ is called the Ricci operator and $\boldsymbol{\partial}\underset{\mathbf{g}}{\cdot}\boldsymbol{\partial} = \square$ is the covariant D' Alem-

[11] This result can be used to show, as stated in the begining of the article that $\mathcal{L}_{g}^{M}$ is invariant under local Lorentz transformations. Indeed, we have that $\mathcal{L}_{g}^{M} = d(\mathtt{h}^{*-1}(\mathfrak{g}^{\mathbf{a}}\wedge \underset{\eta}{\star} d\mathfrak{g}_{\mathbf{a}})) + R\tau_{\eta}$, which is manifestly invariant under the local action of the Lorentz group, since R is a scalar function and $\tau_{\eta}' = \mathfrak{g}'^{0}\wedge\mathfrak{g}'^{1}\wedge\mathfrak{g}'^{2}\wedge\mathfrak{g}'^{3} = u\mathfrak{g}^{0}u^{-1}u\mathfrak{g}^{1}u^{-1}u\mathfrak{g}^{2}u^{-1}u\mathfrak{g}^{3}u^{-1} = u\tau_{\eta}u^{-1} = \tau_{\eta} = \mathfrak{g}^{0}\wedge\mathfrak{g}^{1}\wedge\mathfrak{g}^{2}\wedge\mathfrak{g}^{3}$, for any $u \in \sec \mathrm{Spin}_{1,3}^{e}(M,\eta) \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$.

bertian. We have[12] [23, 22]

$$\begin{aligned}\Diamond\theta^{\mathbf{a}} &= (\boldsymbol{\partial}\wedge\boldsymbol{\partial})\theta^{\mathbf{a}} + \Box\theta^{\mathbf{a}} \\ &= \mathcal{R}^{\mathbf{a}} + \Box\theta^{\mathbf{a}} \\ &= \mathcal{T}^{\mathbf{a}} - \frac{1}{2}\mathcal{T}\theta^{\mathbf{a}} + \Box\theta^{\mathbf{a}}, \end{aligned} \tag{44}$$

where $\mathcal{R}^{\mathbf{a}}$ are the Ricci 1-forms (Eq.(31)). Taking into account also that,

$$\Box\theta^{\mathbf{c}} = -\frac{1}{2}\eta^{\mathbf{ab}}M_{\mathbf{d}}{}^{\mathbf{c}}{}_{\mathbf{ab}}\theta^{\mathbf{d}}, \tag{45}$$

where the tensor field $M_{\mathbf{d}}{}^{\mathbf{c}}{}_{\mathbf{ab}}$ is given by [25, 22]

$$M_{\mathbf{d}}{}^{\mathbf{c}}{}_{\mathbf{ab}} = \mathbf{e_a}(L^{\mathbf{c}}_{\mathbf{bd}}) + \mathbf{e_b}(L^{\mathbf{c}}_{\mathbf{ad}}) - L^{\mathbf{c}}_{\mathbf{ak}}L^{\mathbf{k}}_{\mathbf{bd}} - L^{\mathbf{c}}_{\mathbf{bk}}L^{\mathbf{k}}_{\mathbf{ad}} - (L^{\mathbf{k}}_{\mathbf{ab}\cdot} + L^{\mathbf{k}}_{\mathbf{ba}})L^{\mathbf{c}}_{\mathbf{kd}}, \tag{46}$$

with $\nabla_{\mathbf{e^a}}\theta^{\mathbf{b}} = -L^{\mathbf{b}}_{\mathbf{ac}}\theta^{\mathbf{c}}$ and putting moreover,

$$\mathtt{h}^{*-1}\mathcal{T}^{\mathbf{a}} = \mathcal{T}^{\mathbf{a}}_{M},$$

we finally have

$$\underset{\eta}{\delta}\mathfrak{F}^{\mathbf{a}} = -\mathfrak{J}^{\mathbf{a}}, \tag{47}$$

with

$$\mathfrak{J}^{\mathbf{a}} = \mathcal{T}^{\mathbf{a}}_{M} - \frac{1}{2}\mathcal{T}\mathfrak{g}^{\mathbf{a}} + \frac{1}{2}\eta^{\mathbf{lb}}M_{\mathbf{d}}{}^{\mathbf{a}}{}_{\mathbf{lb}}\mathfrak{g}^{\mathbf{d}} - d\underset{\eta}{\delta}\mathfrak{g}^{\mathbf{a}} \tag{48}$$

The currents $\mathfrak{J}^{\mathbf{a}}$ are conserved, i.e., $\underset{\eta}{\delta}\ \mathfrak{J}^{\mathbf{a}} = 0$ and express the energy-momentum conservation law for the system composed by the gravitational and matter fields. In particular, from Eq.(48) we are tempted to call

$$t^{\mathbf{a}}_{g} = \frac{1}{2}\eta^{\mathbf{lb}}M_{\mathbf{d}}{}^{\mathbf{a}}{}_{\mathbf{lb}}\mathfrak{g}^{\mathbf{d}} - d\underset{\eta}{\delta}\mathfrak{g}^{\mathbf{a}} \tag{49}$$

the true energy momentum 1-forms of the gravitational field and $-\frac{1}{2}\mathcal{T}\mathfrak{g}^{\mathbf{a}}$ the interaction energy-momentum 1-forms. This will be investigate further in another publication.

Remark 3 *Note that* $\mathfrak{F}^{\mathbf{a}} = d\mathfrak{g}^{\mathbf{a}} = \mathtt{h}^{*-1}d\theta^{\mathbf{a}}$*. In the teleparallel equivalent of GRT (see below) the Lorentzian manifold* $(M, \mathbf{g})$ *is equipped with a teleparallel connection such that the torsion* 2*-forms are* $\Theta^{\mathbf{a}} = d\theta^{\mathbf{a}}$*. Then,* Eq.(47) *can be used to write an equation for* $\underset{\mathbf{g}}{\delta}\Theta^{\mathbf{a}}$*, which the reader may find without difficulties.*

[12] Eq.(44) permits the comparison of the Lagrange multiplier $\lambda(x)$ appearing in the theory presented in [12, 13, 14] (where the field equations are $\Diamond\theta^{\mathbf{a}} + \lambda(x)\theta^{\mathbf{a}} = 0$) with its value in General Relativity.

5 Genuine Energy-Momentum Conservation Law

Once we showed that it is possible to express Einstein's gravitational equations as the equations of physical fields $\mathfrak{g}^{\mathbf{a}}$ in Minkowski spacetime we look for the well known[13] form of Einstein's equations in terms of the superpotentials $\mathcal{S}^{\mathbf{a}}$, and which we write here as

$$-d\underset{\mathbf{g}}{\star}\mathcal{S}^{\mathbf{a}} = \underset{\mathbf{g}}{\star}\mathcal{T}^{\mathbf{a}} + \underset{\mathbf{g}}{\star}t^{\mathbf{a}}, \tag{50}$$

with

$$\underset{\mathbf{g}}{\star}t^{\mathbf{c}} = \frac{\partial\mathcal{L}_g}{\partial\theta_a} = -\frac{1}{2}\boldsymbol{\omega}_{\mathbf{ab}} \wedge [\boldsymbol{\omega}_{\mathbf{d}}^{\mathbf{c}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) + \boldsymbol{\omega}_{\mathbf{d}}^{\mathbf{b}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{d}} \wedge \theta^{\mathbf{c}})] \in \sec \bigwedge^3 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g})$$

$$\underset{\mathbf{g}}{\star}\mathcal{S}^{\mathbf{c}} = \frac{\partial\mathcal{L}_g}{\partial d\theta_a} = \frac{1}{2}\boldsymbol{\omega}_{\mathbf{ab}} \wedge \underset{\mathbf{g}}{\star}(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}}) \in \sec \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(T^*M, \mathtt{g}), \tag{51}$$

and where $\boldsymbol{\omega}_{\mathbf{ab}}$ is given by Eq.(26)

As discussed,e.g., in [?] Eq.(50) does not express any trustful energy-momentum conservation law in a general Lorentzian spacetime. However, it express a trustful energy-momentum conservation law in Minkowski spacetime, since it is equivalent (as the reader may verify) to

$$-dh^{-1}\star h\mathcal{S}^{\mathbf{a}} = h^{-1}\star h\mathcal{T}^{\mathbf{a}} + h^{-1}\star ht^{\mathbf{a}}, \tag{52}$$

where in Eq.(52),

$$\star \tag{53}$$

is the Hodge dual relative to the Minkowski metric $\boldsymbol{\eta} = \eta_{\mathbf{ab}}\vartheta^{\mathbf{a}} \otimes \vartheta^{\mathbf{b}}$.

5.1 Mass of the Graviton

In the Lagrangian given by Eq.(23) the mass of the graviton is supposed to be zero. A non null mass m requires an extra term in the Lagrangian. As an example, consider the Lagrangian density

$$\mathcal{L}'_g = -\frac{1}{2}d\theta^{\mathbf{a}}\wedge\underset{\mathbf{g}}{\star}d\theta_{\mathbf{a}} + \frac{1}{2}\underset{\mathbf{g}}{\delta}\theta^{\mathbf{a}}\wedge\underset{\mathbf{g}}{\star}\underset{\mathbf{g}}{\delta}\theta_{\mathbf{a}} + \frac{1}{4}(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}})\wedge\underset{\mathbf{g}}{\star}\left(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}}\right) + \frac{1}{2}m^2\theta_{\mathbf{a}}\wedge\underset{\mathbf{g}}{\star}\theta^{\mathbf{a}} \tag{54}$$

With the extra term the equations for the gravitational field, for the $\mathcal{S}^{\mathbf{a}}$ result in

$$-d\underset{\mathbf{g}}{\star}\mathcal{S}^{\mathbf{a}} = \underset{\mathbf{g}}{\star}\mathcal{T}^{\mathbf{a}} + \underset{\mathbf{g}}{\star}t^{\mathbf{a}} + m^2\underset{\mathbf{g}}{\star}\theta^{\mathbf{a}}, \tag{55}$$

from where we get

$$\underset{\mathbf{g}}{\delta}(\mathcal{T}^{\mathbf{a}} + t^{\mathbf{a}}) = -m^2\underset{\mathbf{g}}{\delta}\theta^{\mathbf{a}} \tag{56}$$

If we impose the gauge $\underset{\mathbf{g}}{\delta}\theta^{\mathbf{a}} = 0$, which is analogous to the Lorenz gauge in electrodynamics, Eq.(56) becomes

$$\underset{\mathbf{g}}{\delta}(\mathcal{T}^{\mathbf{a}} + t^{\mathbf{a}}) = 0, \tag{57}$$

[13] See, e.g.,[22, 27].

which is the same equation valid in the case $m = 0$!

There are other possibilities of having a non null graviton mass, as, e.g., in Logunov's theory [15, 16], which we do not discuss here[14].

6 Teleparallel Geometry

Before closing this paper we briefly recall our discussion in [24] where it was observed that recently some people [2] think to have find a valid way of formulating a genuine energy-momentum conservation law in a theory that is claimed to be (and indeed, it is) equivalent to general relativity. In that theory, the so-called *teleparallel* equivalent of General Relativity theory [17], spacetime is teleparallel (or Weintzböck), i.e., has a metric compatible connection with non zero torsion and with null curvature[15]. We showed in [24] that the claim of [2] must be qualified. Here, our objective is only to show that the teleparallel theory is a possible trivial interpretation of our formalism. Indeed, the structure of the teleparallel equivalent of GRT as formulated, e.g., by [17] or [2] consists in nothing more than a trivial introduction of: (i) a bilinear form (a deformed metric tensor) $\mathbf{g} = \eta_{\mathbf{ab}}\theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}$ and (ii) a teleparallel connection in the manifold $M \simeq \mathbb{R}^4$ of Minkowski spacetime structure. Indeed, taking advantage of the the discussion of the previous sections, we can present that theory with a cosmological constant term as follows. Start with $\mathcal{L}'_g$ (Eq.(54)) and write it (after some algebraic manipulations) as

$$
\begin{aligned}
\mathcal{L}'_g &= -\frac{1}{2} d\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}\left[d\theta_{\mathbf{a}} - \theta_{\mathbf{a}} \wedge (\theta_{\mathbf{b}} \underset{\mathbf{g}}{\lrcorner} d\theta_{\mathbf{b}}) + \frac{1}{2}\underset{\mathbf{g}}{\star}\left(\theta_{\mathbf{a}} \wedge \star(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}})\right)\right] + \frac{1}{2} m^2 \theta_{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}\theta^{\mathbf{a}} \\
&= -\frac{1}{2} d\theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}({}^{(1)}d\theta_{\mathbf{a}} - 2{}^{(2)}d\theta_{\mathbf{a}} - \frac{1}{2}\, {}^{(3)}d\theta_{\mathbf{a}}) + \frac{1}{2} m^2 \theta_{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}\theta^{\mathbf{a}},
\end{aligned}
\tag{58}
$$

where

$$
\begin{aligned}
d\theta^{\mathbf{a}} &= {}^{(1)}d\theta^{\mathbf{a}} + {}^{(2)}d\theta^{\mathbf{a}} + {}^{(3)}d\theta^{\mathbf{a}}, \\
{}^{(1)}d\theta^{\mathbf{a}} &= d\theta^{\mathbf{a}} - {}^{(2)}d\theta^{\mathbf{a}} - {}^{(3)}d\theta^{\mathbf{a}}, \\
{}^{(2)}d\theta^{\mathbf{a}} &= \frac{1}{3}\theta^{\mathbf{b}} \wedge (\theta_{\mathbf{b}} \underset{\mathbf{g}}{\lrcorner} d\theta_{\mathbf{b}}), \\
{}^{(3)}d\theta^{\mathbf{a}} &= -\frac{1}{3} \star\left(\theta^{\mathbf{b}} \wedge \star(d\theta^{\mathbf{b}} \wedge \theta_{\mathbf{b}})\right).
\end{aligned}
\tag{59}
$$

Next introduce a teleparallel connection by declaring that the cobasis $\{\theta^{\mathbf{a}}\}$ fixes the parallelism, i.e., we define the torsion 2-forms by

$$
\Theta^{\mathbf{a}} := d\theta^{\mathbf{a}}, \tag{60}
$$

[14] We only observe that Lagrangian density of Logunov's theory when written in terms of differential forms is not a very elegant expression.

[15] In fact, formulation of teleparallel equivalence of General Relativity is a subject with a old history. See, e.g., [8].

and $\mathcal{L}_g$ becomes

$$\mathcal{L}_g = -\frac{1}{2}\Theta^{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}\left({}^{(1)}\Theta^{\mathbf{a}} - 2{}^{(2)}\Theta^{\mathbf{a}} - \frac{1}{2}\ {}^{(3)}\Theta^{\mathbf{a}}\right) + \frac{1}{2}m^2\theta_{\mathbf{a}} \wedge \underset{\mathbf{g}}{\star}\theta^{\mathbf{a}}, \tag{61}$$

where ${}^{(1)}\Theta^{\mathbf{a}} ={}^{(1)} d\theta^{\mathbf{a}}$, ${}^{(2)}\Theta^{\mathbf{a}} ={}^{(31)} d\theta^{\mathbf{a}}$ and ${}^{(3)}\Theta^{\mathbf{a}} ={}^{(31)} d\theta^{\mathbf{a}}$, called *tractor* (four components), *axitor* (four components) and *tentor* (sixteen components) are the irreducible components of the tensor torsion under the action of $\mathrm{SO}^e_{1,3}$.

7 Conclusion

In the writing of this paper we have been motivated, first by the desire of having genuine energy-momentum and angular momentum conservations laws for the gravitational and matter fields, and second by some thoughts of Kiehn [10] about the physical vacuum. We thus produced (using the Clifford bundle formalism) a theory where the gravitational field represented by $\mathfrak{F}^{\mathbf{a}} = d\mathfrak{g}^{\mathbf{a}}$ (which are physical fields in the Faraday sense, living in Minkowski spacetime, .like the electromagnetic field) satisfy Maxwell like equations, $d\mathfrak{F}^{\mathbf{a}} = 0$, $\underset{\eta}{\delta}\mathfrak{F}^{\mathbf{a}} = \mathfrak{J}^{\mathbf{a}}$, where the currents $\mathfrak{J}^{\mathbf{a}}$ are given by Eq.(48). Moreover, we showed that when the graviton mass is zero, the gravitational field can be interpreted as creating: **(i)** an effective Lorentzian geometry where probe particles and probe fields move, or **(ii)** an effective teleparallel geometry where probe particles and probe fields move. In such theory there are, of course no exotic topologies, black-holes[16], worm-holes, no possibility for time-machines[17], etc., which according to our opinion are pure science fiction objects. Eventually, many will not like the viewpoint just presented, but we feel that many will become interested in exploiting new ideas presented with nice Mathematics, which may be more close to the way Nature operates.

References

[1] Abrams, L.S., Alternative Space=time for a Point Mass, *Phys. Rev. D 20*, 2474-2479 (1979). [gr-qc/0201044]

[16] There are several interesting articles criticizing the notion that black-holes are predicitons of General Relativity on mathematical and physical grounds, as, e.g.[1, 3, 18, 29]. Also the 'pasticcio' concerning the black-hole information 'paradox' (see, [11, 9]) is an example that the foundations of General Relativity are not well understood as some people would like us to think.

[17] The possibility for time machines arises when closed timelike curves exist in a Lorentzian manifold. Such exotic configurations, it is *said*, already appears in Gödel's universe model. However, a recent thoughtful analysis by Cooperstock and Tieu [4] shows that the old claim is wrong. Authors like, e.g, Davies [5] (which are proposing to build time machines even at home), Gott [7] and Novikov [**?**] are invited to read [4] and find a error in the argument of that authors.

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